Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 4, pp. 781 – 799
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.4.781

SOME RESULTS ON CONTINUED FRACTIONS OF ORDER THIRTY-TWO

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ABSTRACT. Chetry and Saikia (2021) derived four continued fractions of order thirty-two from a general continued fraction identity of Ramanujan, and proved some theta-function and modular identities. In this paper, we prove some new theta-function identities for the four continued fractions and derive partition-theoretic results by using colour partition of integers. We establish general theorems for finding explicit values of the continued fractions by using theta-function identities and give examples. We also obtain some vanishing coefficient results for the continued fractions with the help of dissection formulas.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 30B70, 11F27, 11A55,; 11P84.

KEYWORDS AND PHRASES. continued fraction of order thirty-two, theta-function identities, partition of integer; colour partition, explicit values, vanishing coefficients.

1. Introduction

Throughout the paper, for $|q| < 1$ and any complex number a, we use the notation

$$(a;q)_\infty := \prod_{t=0}^{\infty}\left(1 - aq^t\right). \tag{1}$$

For brevity, we often write

$$(a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty\cdots(a_m;q)_\infty = (a_1,a_2,a_3,\ldots,a_m;q)_\infty.$$

Ramanujan's general theta-function $\mathfrak{f}(a,b)$ [2, p. 34, (18.1)] is defined by

$$\mathfrak{f}(a,b) = \sum_{t=-\infty}^{\infty} a^{t(t+1)/2} b^{t(t-1)/2}, \qquad |ab| < 1. \tag{2}$$

Three important special cases of $\mathfrak{f}(a,b)$ [2, p. 36, Entry 22 (i)-(iii)] are given by

$$\phi(q) := \mathfrak{f}(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{3}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{4}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2} = (q;q)_\infty, \tag{5}$$

respectively. It is also useful to note here that

$$\phi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{6}$$

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Submitted on 4 August 2024.

Also, in terms of $\mathfrak{f}(a,b)$, Jacobi's triple product identity [2, p. 35, Entry 19] can be stated as

$$\mathfrak{f}(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty=(-a,-b,ab;ab)_\infty. \tag{7}$$

One of the Ramanujan's remarkable contributions is in the field of continued fractions. An interesting q-continued fraction recorded by Ramanujan on page 299 of his second notebook [7] is the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ given by

$$\begin{aligned} H(q) &:= q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}=q^{1/2}\frac{\mathfrak{f}(-q,-q^7)}{\mathfrak{f}(-q^3,-q^5)} \\ &= \cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cdots}}}. \end{aligned} \tag{8}$$

It is worth to mention here that $H(q)$ is a continued fraction of order eight. Göllnitz [4] and Gordon [5] independently rediscovered and proved (8). Ramanujan also offered following two theta-function identities [7, p. 299] for $H(q)$:

$$\frac{1}{H(q)}-H(q)=\frac{\phi(q^2)}{q^{1/2}\psi(q^4)} \tag{9}$$

and

$$\frac{1}{H(q)}+H(q)=\frac{\phi(q)}{q^{1/2}\psi(q^4)}. \tag{10}$$

Proofs of (9) and (10) can be found in [2, p. 221]. Baruah and Saikia [1] and Saikia [8] established some general theorems for explicit evaluations of $H(q)$ and evaluated some values.

In 2021, Chetry and Saikia [3] obtained four continued fractions $J_1(q)$, $J_2(q)$, $J_3(q)$ and $J_4(q)$ of order thirty-two, which are given by

$$\begin{aligned} J_1(q) &= q^{3/2}\frac{\mathfrak{f}(-q^5,-q^{27})}{\mathfrak{f}(-q^{11},-q^{21})} \\ &= \cfrac{q^{3/2}(1-q^5)}{(1-q^8)+\cfrac{q^8(1-q^3)(1-q^{13})}{(1-q^8)(1+q^{16})+\cfrac{q^8(1-q^{19})(1-q^{29})}{(1-q^8)(1+q^{32})+\cdots}}}, \end{aligned} \tag{11}$$

$$\begin{aligned} J_2(q) &= q^{1/2}\frac{\mathfrak{f}(-q^7,-q^{25})}{\mathfrak{f}(-q^{9},-q^{23})} \\ &= \cfrac{q^{1/2}(1-q^7)}{(1-q^8)+\cfrac{q^8(1-q)(1-q^{15})}{(1-q^8)(1+q^{16})+\cfrac{q^8(1-q^{17})(1-q^{31})}{(1-q^8)(1+q^{32})+\cdots}}}, \end{aligned} \tag{12}$$

$$J_3(q) = q^{5/2}\frac{\mathfrak{f}(-q^3,-q^{29})}{\mathfrak{f}(-q^{13},-q^{19})} \tag{13}$$

$$= \cfrac{q^{5/2}(1-q^3)}{(1-q^8)+\cfrac{q^8(1-q^5)(1-q^{11})}{(1-q^8)(1+q^{16})+\cfrac{q^8(1-q^{21})(1-q^{27})}{(1-q^8)(1+q^{32})+\cdots}}}$$

and

$$J_4(q) = q^{7/2}\frac{\mathfrak{f}(-q,-q^{31})}{\mathfrak{f}(-q^{15},-q^{17})} \tag{14}$$

$$= \cfrac{q^{7/2}(1-q)}{(1-q^8)+\cfrac{q^8(1-q^7)(1-q^{9})}{(1-q^8)(1+q^{16})+\cfrac{q^8(1-q^{23})(1-q^{25})}{(1-q^8)(1+q^{32})+\cdots}}}.$$

They also established following theta-function and modular identities [3, Theorem 2.1(i)-(v)] for the continued fractions $J_1(q)$, $J_2(q)$, $J_3(q)$ and $J_4(q)$:

$$\frac{1}{J_1(q)} - J_1(q) = \frac{\mathfrak{f}(-q^3,-q^{13})\phi(q^8)}{q^{3/2}\mathfrak{f}(-q^{11},-q^{21})\mathfrak{f}(-q^5,-q^{27})}, \tag{15}$$

$$\frac{1}{J_2(q)} - J_2(q) = \frac{\mathfrak{f}(-q,-q^{15})\phi(q^8)}{q^{1/2}\mathfrak{f}(-q^7,-q^{25})\mathfrak{f}(-q^9,-q^{23})}, \tag{16}$$

$$\frac{1}{J_3(q)} - J_3(q) = \frac{\mathfrak{f}(-q^5,-q^{11})\phi(q^8)}{q^{5/2}\mathfrak{f}(-q^3,-q^{29})\mathfrak{f}(-q^{13},-q^{19})}, \tag{17}$$

$$\frac{1}{J_4(q)} - J_4(q) = \frac{\mathfrak{f}(-q^7,-q^{9})\phi(q^8)}{q^{7/2}\mathfrak{f}(-q,-q^{31})\mathfrak{f}(-q^{15},-q^{17})} \tag{18}$$

and

$$\left(\frac{1}{J_1(q)} - J_1(q)\right)\left(\frac{1}{J_3(q)} - J_3(q)\right) = \left(\frac{1}{J_2(q)} - J_2(q)\right)\left(\frac{1}{J_4(q)} - J_4(q)\right). \tag{19}$$

By proving dissection formulas, Chetry and Saikia [3] showed that, if

$$J_1^*(q) = q^{-3/2}J_1(q) = \frac{\mathfrak{f}(-q^5,-q^{27})}{\mathfrak{f}(-q^{11},-q^{21})} = \sum_{n=0}^{\infty} a_n q^n \quad \text{and} \quad \frac{1}{J_1^*(q)} = \sum_{n=0}^{\infty} b_n q^n,$$

then

$$a_{16n+14} = 0 \quad \text{and} \quad b_{16n+1} = 0.$$

In this sequel, we establish some new theta-function identities for the continued fractions $J_1(q)$, $J_2(q)$, $J_3(q)$ and $J_4(q)$ in Section 2 of this paper. In Section 3, we obtain partition-theoretic results from the theta-function identities of the continued fractions by using colour partition of integers. Section 4 is devoted to proving general theorems to find explicit values of the four continued fractions. Finally, in Section 5, we obtain some vanishing coefficient results for the continued fractions $J_2(q)$, $J_3(q)$ and $J_4(q)$ with the help of dissection formulas.

2. New theta-function and modular identities

Theorem 2.1. *We have*

(i) $\dfrac{1}{J_1(q)} + J_1(q) = \dfrac{\mathfrak{f}(q^3, q^{13})\phi(-q^8)}{q^{3/2}\mathfrak{f}(-q^5, -q^{11})\psi(q^{16})}$,

(ii) $\dfrac{1}{J_2(q)} + J_2(q) = \dfrac{\mathfrak{f}(q, q^{15})\phi(-q^8)}{q^{1/2}\mathfrak{f}(-q^7, -q^9)\psi(q^{16})}$,

(iii) $\dfrac{1}{J_3(q)} + J_3(q) = \dfrac{\mathfrak{f}(q^5, q^{11})\phi(-q^8)}{q^{5/2}\mathfrak{f}(-q^3, -q^{13})\psi(q^{16})}$,

(iv) $\dfrac{1}{J_4(q)} + J_4(q) = \dfrac{\mathfrak{f}(q^7, q^9)\phi(-q^8)}{q^{7/2}\mathfrak{f}(-q, -q^{15})\psi(q^{16})}$,

(v) $\left(\dfrac{1}{J_3(q)} - J_3(q)\right) - \left(\dfrac{1}{J_1(q)} - J_1(q)\right) = \dfrac{\phi(q^8)\phi(q^4)\left(\phi(q) - \phi(q^2)\right)}{2q^{7/2}\psi(q^{16})\psi(q^8)\psi(q^4)}$,

(vi) $\left(\dfrac{1}{J_4(q)} - J_4(q)\right) - \left(\dfrac{1}{J_2(q)} - J_2(q)\right) = \dfrac{\phi(q^8)\phi(q^4)\left(\phi(q) + \phi(q^2)\right)}{2q^{7/2}\psi(q^{16})\psi(q^8)\psi(q^4)}$,

(vii) $\left(\dfrac{1}{J_3(q)} - J_3(q)\right) + \left(\dfrac{1}{J_1(q)} - J_1(q)\right) = \dfrac{\phi^2(-q^{16})\phi(-q^4)\mathfrak{f}(-q^2, -q^{14})}{q^{5/2}\psi(q^{16})\psi(q^8)\psi(q^4)\psi(-q)}$,

(viii) $\left(\dfrac{1}{J_4(q)} - J_4(q)\right) + \left(\dfrac{1}{J_2(q)} - J_2(q)\right) = \dfrac{\phi^2(-q^{16})\phi(-q^4)\mathfrak{f}(-q^6, -q^{10})}{q^{7/2}\psi(q^{16})\psi(q^8)\psi(q^4)\psi(-q)}$.

Proof. From (11), we obtain

(20) $$\frac{1}{\sqrt{J_1(q)}} + \sqrt{J_1(q)} = \frac{\mathfrak{f}(-q^{11}, -q^{21}) + q^{3/2}\mathfrak{f}(-q^5, -q^{27})}{\sqrt{q^{3/2}\mathfrak{f}(-q^5, -q^{27})\mathfrak{f}(-q^{11}, -q^{21})}}.$$

From [2, p. 46, Entry 30 (ii) and (iii)], we note that

(21) $$\mathfrak{f}(a, b) = \mathfrak{f}(a^3 b, ab^3) + a\mathfrak{f}(b/a, a^5 b^3).$$

Setting $a = q^{3/2}$ and $b = -q^{13/2}$ in (21), we obtain

(22) $$\mathfrak{f}(q^{3/2}, -q^{13/2}) = \mathfrak{f}(-q^{11}, -q^{21}) + q^{3/2}\mathfrak{f}(-q^5, -q^{27}).$$

Again, from [2, p. 46, Entry 30 (i)], we note that

(23) $$\mathfrak{f}(a, ab^2)\mathfrak{f}(b, a^2 b) = \mathfrak{f}(a, b)\psi(ab).$$

Setting $a = -q^5$ and $b = -q^{11}$ in (23), we obtain

(24) $$\mathfrak{f}(-q^5, -q^{27})\mathfrak{f}(-q^{11}, -q^{21}) = \mathfrak{f}(-q^5, -q^{11})\psi(q^{16}).$$

Employing (22) in (20), we find that

(25) $$\frac{1}{\sqrt{J_1(q)}} + \sqrt{J_1(q)} = \frac{\mathfrak{f}(q^{3/2}, -q^{13/2})}{\sqrt{q^{3/2}\mathfrak{f}(-q^5, -q^{11})\psi(q^{16})}}.$$

Squaring (25), we obtain

(26) $$\frac{1}{J_1(q)} + J_1(q) = \frac{\mathfrak{f}^2(q^{3/2}, -q^{13/2})}{q^{3/2}\mathfrak{f}(-q^5, -q^{11})\psi(q^{16})} - 2.$$

From [2, p. 46, Entry 30 (v),(vi)], we note that

(27) $$\mathfrak{f}^2(a, b) = \mathfrak{f}(a^2, b^2)\phi(ab) + 2a\mathfrak{f}(b/a, a^3 b)\psi(a^2 b^2).$$

Setting $a = q^{3/2}$ and $b = -q^{13/2}$ in (27), we obtain

$$\mathfrak{f}^2(q^{3/2}, -q^{13/2}) = \mathfrak{f}(q^3, q^{13})\phi(-q^8) + 2q^{3/2}\mathfrak{f}(-q^5, -q^{11})\psi(q^{16}). \tag{28}$$

Employing (28) in (26), we arrive at (i). Similarly, we can prove (ii)-(iv). Setting $a = -q^3$ and $b = -q^{13}$ in (23), we obtain

$$\mathfrak{f}(-q^3, -q^{29})\mathfrak{f}(-q^{13}, -q^{19}) = \mathfrak{f}(-q^3, -q^{13})\psi(q^{16}). \tag{29}$$

Rewriting (15) and (17) using (24) and (29), we have

$$\frac{1}{J_1(q)} - J_1(q) = \frac{\mathfrak{f}(-q^3, -q^{13})\phi(q^8)}{q^{3/2}\mathfrak{f}(-q^5, -q^{11})\psi(q^{16})} \tag{30}$$

and

$$\frac{1}{J_3(q)} - J_3(q) = \frac{\mathfrak{f}(-q^5, -q^{11})\phi(-q^8)}{q^{5/2}\mathfrak{f}(-q^3, -q^{13})\psi(q^{16})}, \tag{31}$$

respectively. From (30) and (31), we have

$$\left(\frac{1}{J_3(q)} - J_3(q)\right) - \left(\frac{1}{J_1(q)} - J_1(q)\right) = \frac{\phi(q^8)\left\{\mathfrak{f}^2(-q^5, -q^{11}) - q\mathfrak{f}^2(-q^3, -q^{13})\right\}}{q^{5/2}\psi(q^{16})\mathfrak{f}(-q^3, -q^{13})\mathfrak{f}(-q^5, -q^{11})}. \tag{32}$$

Setting $a = -q^5, b = -q^{11}$ and $a = -q^3, b = -q^{13}$ in (27), we obtain

$$\mathfrak{f}^2(q^5, -q^{11}) = \mathfrak{f}(q^{10}, q^{22})\phi(q^{16}) - 2q^5\mathfrak{f}(q^6, q^{26})\psi(q^{32}) \tag{33}$$

and

$$\mathfrak{f}^2(q^3, -q^{13}) = \mathfrak{f}(q^6, q^{26})\phi(q^{16}) - 2q^3\mathfrak{f}(q^{10}, q^{22})\psi(q^{32}), \tag{34}$$

respectively. Setting $a = -q^3$ and $b = -q^5$ in (23), we obtain

$$\mathfrak{f}(-q^3, -q^{13})\mathfrak{f}(-q^5, -q^{11}) = \mathfrak{f}(-q^3, -q^5)\psi(q^8). \tag{35}$$

Employing (33)-(35) in (32), we obtain

$$\left(\frac{1}{J_3(q)} - J_3(q)\right) - \left(\frac{1}{J_1(q)} - J_1(q)\right) = \frac{\phi(q^8)\left\{\left(\mathfrak{f}(q^{10}, q^{22}) - q\mathfrak{f}(q^6, q^{26})\right)\left(\phi(q^{16}) + 2q^4\psi(q^{32})\right)\right\}}{q^{5/2}\psi(q^{16})\mathfrak{f}(-q^3, -q^5)\psi(q^8)}. \tag{36}$$

Setting $a = -q$ and $b = -q^7$ in (21), we obtain

$$\mathfrak{f}(-q, -q^7) = \mathfrak{f}(q^{10}, q^{22}) - q\mathfrak{f}(q^6, q^{26}). \tag{37}$$

From [2, p. 40, Entry 25 (i) and (ii)], we note that

$$\phi(q^4) + 2q\psi(q^8) = \phi(q) \tag{38}$$

and

$$\phi(q^4) - 2q\psi(q^8) = \phi(-q). \tag{39}$$

Replacing q by q^4 in (38), we obtain

$$\phi(q^{16}) + 2q^4\psi(q^{32}) = \phi(q^4). \tag{40}$$

Employing (37) and (40) in (36), we have

$$\left(\frac{1}{J_3(q)} - J_3(q)\right) - \left(\frac{1}{J_1(q)} - J_1(q)\right) = \frac{\phi(q^8)\phi(q^4)\mathfrak{f}^2(-q,-q^7)}{q^{5/2}\psi(q^{16})\psi(q^8)\mathfrak{f}(-q^3,-q^5)\mathfrak{f}(-q,-q^7)}. \tag{41}$$

From [2, p. 51] (with q by $-q$), we note that

$$\phi(q) + \phi(q^2) = \frac{2\mathfrak{f}^2(-q^3,-q^5)}{\psi(-q)} \tag{42}$$

and

$$\phi(q) - \phi(q^2) = \frac{2q\mathfrak{f}^2(-q,-q^7)}{\psi(-q)}. \tag{43}$$

Setting $a = -q$, $b = -q^3$ in (23), we obtain

$$\mathfrak{f}(-q,-q^7)\mathfrak{f}(-q^3,-q^5) = \mathfrak{f}(-q,-q^3)\psi(q^4) = \psi(-q)\psi(q^4). \tag{44}$$

Employing (43) and (44) in (41), we arrive at (v). Proofs of (vi)-(viii) are similar to the proof of (v), so we omit. □

Theorem 2.2. *For any positive integer n, we have*

$$(i)\ J_1^n(q)J_1^n(-q) = \begin{cases} J_1^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -J_1^n(q^2), & if \quad n \equiv 2 \pmod 4, \end{cases}$$

$$(ii)\ J_2^n(q)J_2^n(-q) = \begin{cases} J_2^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -J_2^n(q^2), & if \quad n \equiv 2 \pmod 4, \end{cases}$$

$$(iii)\ J_3^n(q)J_3^n(-q) = \begin{cases} J_3^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -J_3^n(q^2), & if \quad n \equiv 2 \pmod 4, \end{cases}$$

$$(iv)\ J_4^n(q)J_4^n(-q) = \begin{cases} J_4^n(q^2), & if \quad n \equiv 0 \pmod 4 \\ -J_4^n(q^2), & if \quad n \equiv 2 \pmod 4. \end{cases}$$

Proof. From (11), we note that

$$J_1^n(q)J_1^n(-q) = (-1)^{3n/2}q^{3n}\frac{\mathfrak{f}^n(-q^5,-q^{27})}{\mathfrak{f}^n(-q^{11},-q^{21})} \times \frac{\mathfrak{f}^n(q^5,q^{27})}{\mathfrak{f}^n(q^{11},q^{21})}. \tag{45}$$

Setting $a = q^5, b = q^{27}$ and $a = q^{11}, b = q^{21}$ in (88), we find that

$$\mathfrak{f}(q^5,q^{27})\mathfrak{f}(-q^5,-q^{27}) = \mathfrak{f}(-q^{10},-q^{54})\phi(-q^{32}) \tag{46}$$

and

$$\mathfrak{f}(q^{11},q^{21})\mathfrak{f}(-q^{11},-q^{21}) = \mathfrak{f}(-q^{22},-q^{42})\phi(-q^{32}), \tag{47}$$

respectively. Employing (46) and (47) in (45), we obtain

$$J_1^n(q)J_1^n(-q) = (-1)^{3n/2}q^{3n}\frac{\mathfrak{f}^n(-q^{10},-q^{54})}{\mathfrak{f}^n(-q^{22},-q^{42})} = (-1)^{3n/2}J_1^n(q^2). \tag{48}$$

Noting the fact that $3n/2$ is even if $n \equiv 0 \pmod 4$ and odd if $n \equiv 2 \pmod 4$ in (48), we complete the proof of (i). Proofs of (ii)-(iv) are identical to the proof of (i), so we omit. □

3. Partition-theoretic results

At first, we define partition and colour partition of a positive integer. A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 3$ has three partitions, namely,

$$3, \quad 2+1, \quad 1+1+1.$$

If $p(n)$ denote the number of partitions of n, then $p(3) = 3$. The generating function for $p(n)$ due to Euler is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{49}$$

A part in a partition of n is said to have r colours if each part has r copies and all of them are viewed as distinct objects. For any positive integers n and r, let $p_r(n)$ denote the number of partitions of n with each part having r distinct colours. For example, if each part in the partitions of 3 has two colours, say white (indicated by the suffix w) and blue (indicated by the suffix b), then the number of two colour partitions of 3 is 10 (that is, $p_2(3) = 10$), namely 3_w, 3_b, $2_w + 1_w$, $2_w + 1_b$, $2_b + 1_b$, $2_b + 1_w$, $1_w + 1_w + 1_w$, $1_w + 1_w + 1_b$, $1_w + 1_b + 1_b$, $1_b + 1_b + 1_b$.

The generating function of $p_r(n)$ is given by

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{(q;q)_\infty^r}. \tag{50}$$

Also, for positive integers s, m and r, the quotient

$$\frac{1}{(q^s;q^m)_\infty^r} \tag{51}$$

is the generating function of the number of partitions of n with parts congruent to s modulo m and each parts having r distinct colours. For example,

$$\frac{1}{(q^{s_1};q^m)_\infty^\ell (q^{s_2};q^m)_\infty^\ell} = \frac{1}{(q^{s_1},q^{s_2};q^m)_\infty^\ell} \tag{52}$$

is the generating function of the number of partitions with parts congruent to s_1 or s_2 modulo m and each part has ℓ distinct colours.

In this section, for convenience we will use the notation

$$(q^{r\pm};q^t) := (q^r, q^{t-r};q^t)_\infty, \tag{53}$$

where r and t are positive integers and $r < t$.

Theorem 3.1. *Let $\mathcal{C}_1(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm5, \pm13$ or $\pm16 \pmod{32}$ such that the parts congruent to ±5 and $\pm16 \pmod{32}$ have 2 colours. Let $\mathcal{C}_2(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm11, \pm13$ or $\pm16 \pmod{32}$ such that parts congruent to ±11 and $\pm16 \pmod{32}$ have 2 colours. Let $\mathcal{C}_3(n)$ denote the number of partitions of n into parts congruent to $\pm5, \pm8$ or $\pm11 \pmod{32}$ with 2 colours. Then for any integer $n \geq 3$,*

$$\mathcal{C}_1(n) - \mathcal{C}_2(n-3) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (3), (7) and (11) in (15) and simplifying, we obtain

$$\frac{(q^{11\pm};q^{32})_\infty}{q^{3/2}(q^{5\pm};q^{32})_\infty}-q^{3/2}\frac{(q^{5\pm};q^{32})_\infty}{(q^{11\pm};q^{32})_\infty}-\frac{(q^{3\pm},q^{13\pm};q^{32})_\infty(q^{16\pm};q^{32})^2_\infty}{q^{3/2}(q^{5\pm},q^{11\pm};q^{32})_\infty(q^{8\pm};q^{32})^2_\infty}=0. \tag{54}$$

Dividing (54) by $(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{16\pm},q^{32})^2_\infty$, we obtain

$$\frac{1}{(q^{3\pm,13\pm};q^{32})_\infty(q^{5\pm,16\pm};q^{32})^2_\infty}-\frac{q^3}{(q^{3\pm,13\pm};q^{32})_\infty(q^{11\pm,16\pm};q^{32})^2_\infty}-\frac{1}{(q^{5\pm,8\pm,11\pm};q^{32})^2_\infty}=0. \tag{55}$$

The above quotients of (55) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (55) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n-q^3\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n-\sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n=0, \tag{56}$$

where we set $\mathcal{C}_1(0)=\mathcal{C}_2(0)=\mathcal{C}_3(0)=1$. Equating coefficients of q^n on both sides of (56), we arrive at the desired result. □

Example:

TABLE 1. The case $n=5$ in Theorem 3.1.

$\mathcal{C}_1(5)=2$	$\mathcal{C}_2(2)=0$	$\mathcal{C}_3(5)=2$
5_r		5_r
5_g		5_g

Theorem 3.2. *Let $\mathcal{C}_1(n)$ denote the number of partitions of n into parts congruent to $\pm1,\pm7,\pm15$ or ±16* (mod 32) *such that the parts congruent to ±7 and ±16* (mod 32) *have 2 colours. Let $\mathcal{C}_2(n)$ denote the number of partitions of n into parts congruent to $\pm1,\pm9,\pm15$ or ±16* (mod 32) *such that parts congruent to ±9 and ±16* (mod 32) *have 2 colours. Let $\mathcal{C}_3(n)$ denote the number of partitions of n into parts congruent to $\pm7,\pm8$ or ±9* (mod 32) *with 2 colours. Then for any integer $n\geq1$,*

$$\mathcal{C}_1(n)-\mathcal{C}_2(n-1)-\mathcal{C}_3(n)=0.$$

Proof. Employing (3), (7) and (12) in (16) and simplifying, we obtain

$$\frac{(q^{9\pm};q^{32})_\infty}{q^{1/2}(q^{7\pm};q^{32})_\infty}-q^{1/2}\frac{(q^{7\pm};q^{32})_\infty}{(q^{9\pm};q^{32})_\infty}-\frac{(q^{1\pm},q^{15\pm};q^{32})_\infty(q^{16\pm};q^{32})^2_\infty}{q^{1/2}(q^{7\pm},q^{9\pm};q^{32})_\infty(q^{8\pm};q^{32})^2_\infty}=0. \tag{57}$$

Dividing (57) by $(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{16\pm},q^{32})^2_\infty$, we obtain

$$\frac{1}{(q^{1\pm,15\pm};q^{32})_\infty(q^{7\pm,16\pm};q^{32})^2_\infty}-\frac{q}{(q^{1\pm,15\pm};q^{32})_\infty(q^{9\pm,16\pm};q^{32})^2_\infty}-\frac{1}{(q^{7\pm,8\pm,9\pm};q^{32})^2_\infty}=0. \tag{58}$$

The quotients of (58) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (58) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n-q\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n-\sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n=0, \tag{59}$$

where we set $\mathcal{C}_1(0)=\mathcal{C}_2(0)=\mathcal{C}_3(0)=1$. Equating coefficients of q^n on both sides of (59), we arrive at the desired result. □

Example:

TABLE 2. The case $n = 7$ in Theorem 3.2.

$\mathcal{C}_1(7) = 3$	$\mathcal{C}_2(6) = 1$	$\mathcal{C}_3(7) = 2$
7_r	$1+1+1+1+1+1$	7_r
7_g		7_g
$1+1+1+1+1+1+1$		

Theorem 3.3. *Let $\mathcal{C}_1(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm5, \pm11$ or ±16 (mod 32) such that the parts congruent to ±3 and ±16 (mod 32) have 2 colours. Let $\mathcal{C}_2(n)$ denote the number of partitions of n into parts congruent to $\pm5, \pm11, \pm13$ or ±16 (mod 32) such that parts congruent to ±13 and ±16 (mod 32) have 2 colours. Let $\mathcal{C}_3(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm8$ or ±13 (mod 32) with 2 colours. Then for any integer $n \geq 5$,*

$$\mathcal{C}_1(n) - \mathcal{C}_2(n-5) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (3), (7) and (13) in (17), we obtain

$$\frac{(q^{13\pm};q^{32})_\infty}{q^{5/2}(q^{3\pm};q^{32})_\infty} - q^{5/2}\frac{(q^{3\pm};q^{32})_\infty}{(q^{13\pm};q^{32})_\infty} - \frac{(q^{5\pm},q^{11\pm};q^{32})_\infty(q^{16\pm};q^{32})^2_\infty}{q^{5/2}(q^{3\pm},q^{13\pm};q^{32})_\infty(q^{8\pm};q^{32})^2_\infty} = 0. \tag{60}$$

Dividing (60) by $(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{16\pm},q^{32})^2_\infty$, we obtain

$$\frac{1}{(q^{5\pm,11\pm};q^{32})_\infty(q^{3\pm,16\pm};q^{32})^2_\infty} - \frac{q^5}{(q^{5\pm,11\pm};q^{32})_\infty(q^{13\pm,16\pm};q^{32})^2_\infty} - \frac{1}{(q^{3\pm,8\pm,13\pm};q^{32})^2_\infty} = 0. \tag{61}$$

The above quotients of (61) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (61) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n - q^5\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n - \sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n = 0, \tag{62}$$

where we set $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathcal{C}_3(0) = 1$. Equating coefficients of q^n on both sides of (62), we arrive at the desired result. □

Example:

TABLE 3. The case $n = 8$ in Theorem 3.3.

$\mathcal{C}_1(8) = 2$	$\mathcal{C}_2(3) = 0$	$\mathcal{C}_3(8) = 2$
$5+3_r$		8_r
$5+3_g$		8_g

Theorem 3.4. *Let $\mathcal{C}_1(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm7, \pm9$ or ±16 (mod 32) such that the parts congruent to ±1 and ±16 (mod 32) have 2 colours. Let $\mathcal{C}_2(n)$ denote the number of partitions of n into parts congruent to $\pm7, \pm9, \pm15$ or ±16 (mod 32) such that parts congruent to ±16 (mod 32) have*

2 colours. Let $\mathcal{C}_3(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm8$ or $\pm15 \pmod{32}$ with 2 colours. Then for any integer $n \geq 7$,

$$\mathcal{C}_1(n) - \mathcal{C}_2(n-7) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (3), (7) and (14) in (18) and simplifying, we obtain

$$\frac{(q^{15\pm};q^{32})_\infty}{q^{7/2}(q^{1\pm};q^{32})_\infty} - q^{7/2}\frac{(q^{1\pm};q^{32})_\infty}{(q^{15\pm};q^{32})_\infty} - \frac{(q^{7\pm},q^{9\pm};q^{32})_\infty(q^{16\pm};q^{32})^2_\infty}{q^{5/2}(q^{1\pm},q^{15\pm};q^{32})_\infty(q^{8\pm};q^{32})^2_\infty} = 0. \tag{63}$$

Dividing (63) by $(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{16\pm},q^{32})^2_\infty$, we obtain

$$\frac{1}{(q^{7\pm,9\pm};q^{32})_\infty(q^{1\pm,16\pm};q^{32})^2_\infty} - \frac{q^7}{(q^{7\pm,9\pm,15\pm};q^{32})_\infty(q^{16\pm};q^{32})^2_\infty} - \frac{1}{(q^{1\pm,8\pm,15\pm};q^{32})^2_\infty} = 0. \tag{64}$$

The above quotients of (61) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (61) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n - q^7\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n - \sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n = 0, \tag{65}$$

where we set $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathcal{C}_3(0) = 1$. Equating coefficients of q^n on both sides of (65), we arrive at the desired result. □

Example:

TABLE 4. The case $n = 7$ in Theorem 3.4.

$\mathcal{C}_1(7) = 2$	$\mathcal{C}_2(0) = 1$	$\mathcal{C}_3(7) = 1$
7		$1+1+1+1+1+1+1$
1+1+1+1+1+1+1		

Theorem 3.5. *Let $\mathcal{C}_1(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm5 \pm7, \pm11$ or $\pm13 \pmod{32}$ such that the parts congruent to ±1 and $\pm7 \pmod{32}$ have 2 colours. Let $\mathcal{C}_2(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm5 \pm 9, \pm11$ or $\pm 13 \pmod{32}$ such that parts congruent to ±1 and $\pm 9 \pmod{32}$ have 2 colours. Let $\mathcal{C}_3(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm5, \pm7, \pm11, \pm13$ or $\pm 15 \pmod{32}$ such that parts congruent to $\pm7, \pm15 \pmod{32}$ have 2 colours. Let $\mathcal{C}_4(n)$ denote the number of partitions of n into parts congruent to $\pm3, \pm5, \pm9, \pm11, \pm13$ or $\pm15 \pmod{32}$ such that parts congruent to $\pm9, \pm15 \pmod{32}$ have 2 colours. Let $\mathcal{C}_5(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm5, \pm7, \pm9$ or $\pm15 \pmod{32}$ such that parts congruent to $\pm3, \pm5 \pmod{32}$ have 2 colours. Let $\mathcal{C}_6(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm3, \pm7, \pm9, \pm11$ or $\pm 15 \pmod{32}$ such that parts congruent to $\pm3, \pm11 \pmod{32}$ have 2 colours. Let $\mathcal{C}_7(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm5, \pm7, \pm9, \pm13$ or $\pm 15 \pmod{32}$ such that parts congruent to $\pm5, \pm13 \pmod{32}$ have 2 colours. Let $\mathcal{C}_8(n)$ denote the number of partitions of n into parts congruent to $\pm1, \pm7, \pm9, \pm11, \pm13$ or ± 15*

(mod 32) *such that parts congruent to* $\pm 11, \pm 13 \pmod{32}$ *have 2 colours. Then for any integer* $n \geq 8$,

$$\mathcal{C}_1(n) - \mathcal{C}_2(n-1) - \mathcal{C}_3(n-7) + \mathcal{C}_4(n-8) - \mathcal{C}_5(n) + \mathcal{C}_6(n-3) \\ + \mathcal{C}_7(n-5) - \mathcal{C}_8(n-8) = 0.$$

Proof. Employing (3), (7) and (11)-(14) in (19) and simplifying, we obtain

$$\begin{aligned}(66)\qquad &\frac{(q^{9\pm,15\pm};q^{32})_\infty}{q^4(q^{1\pm,7\pm};q^{32})_\infty} - \frac{(q^{7\pm,15\pm};q^{32})_\infty}{q^3(q^{1\pm,9\pm};q^{32})_\infty} - \frac{q^3(q^{1\pm,9\pm};q^{32})_\infty}{(q^{7\pm,15\pm};q^{32})_\infty} \\ &+\frac{q^4(q^{1\pm,7\pm};q^{32})_\infty}{(q^{9\pm,15\pm};q^{32})_\infty} - \frac{(q^{11\pm,13\pm};q^{32})_\infty}{q^4(q^{3\pm,5\pm};q^{32})_\infty} + \frac{(q^{5\pm,13\pm};q^{32})_\infty}{q(q^{3\pm,11\pm};q^{32})_\infty} \\ &+\frac{q(q^{3\pm,11\pm};q^{32})_\infty}{(q^{5\pm,13\pm};q^{32})_\infty} - \frac{q^4(q^{3\pm,5\pm};q^{32})_\infty}{(q^{11\pm,13\pm};q^{32})_\infty} = 0.\end{aligned}$$

Dividing (66) by $(q^{1\pm,3\pm,5\pm,7\pm,9\pm,11\pm,13\pm,15\pm};q^{32})_\infty$, we obtain

$$\begin{aligned}(67)\qquad &\frac{1}{(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{1\pm,7\pm};q^{32})_\infty^2} - \frac{q}{(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{1\pm,9\pm};q^{32})_\infty^2} \\ &-\frac{q^7}{(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{7\pm,15\pm};q^{32})_\infty^2} + \frac{q^8}{(q^{3\pm,5\pm,11\pm,13\pm};q^{32})_\infty(q^{9\pm,15\pm};q^{32})_\infty^2} \\ &-\frac{1}{(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{3\pm,5\pm};q^{32})_\infty^2} + \frac{q^3}{(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{3\pm,11\pm};q^{32})_\infty^2} \\ &+\frac{q^5}{(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{5\pm,13\pm};q^{32})_\infty^2} - \frac{q^8}{(q^{1\pm,7\pm,9\pm,15\pm};q^{32})_\infty(q^{11\pm,13\pm};q^{32})_\infty^2} = 0.\end{aligned}$$

The above quotients of (67) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$, $\mathcal{C}_3(n)$, $\mathcal{C}_4(n)$, $\mathcal{C}_5(n)$, $\mathcal{C}_6(n)$, $\mathcal{C}_7(n)$ and $\mathcal{C}_8(n)$, respectively. Hence, (67) is equivalent to

$$\begin{aligned}(68)\qquad &\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n - q\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n - q^7\sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n + q^8\sum_{n=0}^{\infty}\mathcal{C}_4(n)q^n \\ &-\sum_{n=0}^{\infty}\mathcal{C}_5(n)q^n + q^3\sum_{n=0}^{\infty}\mathcal{C}_6(n)q^n + q^5\sum_{n=0}^{\infty}\mathcal{C}_7(n)q^n - q^8\sum_{n=0}^{\infty}\mathcal{C}_8(n)q^n = 0,\end{aligned}$$

where we set $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathcal{C}_3(0) = \mathcal{C}_4(0) = \mathcal{C}_5(0) = \mathcal{C}_6(0) = \mathcal{C}_7(0) = \mathcal{C}_8(0) = 1$. Equating coefficients of q^n on both sides of (68), we arrive at the desired result. □

Example: To illustrate Theorem 3.5 consider the case $n = 8$. By enumerating the relevant partitions, once can see that $\mathcal{C}_1(8) = 27$, $\mathcal{C}_2(7) = 18$, $\mathcal{C}_3(1) = 0$, $\mathcal{C}_4(0) = 1$, $\mathcal{C}_5(8) = 13$, $\mathcal{C}_6(5) = 3$, $\mathcal{C}_7(3) = 1$ and $\mathcal{C}_8(0) = 1$.

Theorem 3.6. *Let* $\mathcal{C}_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 5, \pm 6, \pm 8$ *or* $\pm 16 \pmod{32}$ *such that the parts congruent to* ± 5 *and* $\pm 8 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_2(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 6, \pm 8, \pm 11$ *or* $\pm 16 \pmod{32}$ *such that parts congruent to* ± 8 *and* $\pm 11 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 3, \pm 5, \pm 11$ *or* $\pm 13 \pmod{32}$ *such that the parts congruent to* ± 5 *and* $\pm 11 \pmod{32}$ *have 2 colours. Then for any integer* $n \geq 3$,

$$\mathcal{C}_1(n) + \mathcal{C}_2(n-3) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (4), (6), (7) and (11) in Theorem 2.1 (i) and employing the same procedure, we obtain

$$\frac{1}{(q^{6\pm,16\pm};q^{32})_\infty(q^{5\pm,8\pm};q^{32})^2_\infty}+\frac{q^3}{(q^{6\pm,16\pm};q^{32})_\infty(q^{8\pm,11\pm};q^{32})^2_\infty}-\frac{1}{(q^{3\pm,13\pm};q^{32})_\infty(q^{5\pm,11\pm};q^{32})^2_\infty}=0. \tag{69}$$

The above quotients of (69) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (69) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n+q^3\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n-\sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n=0, \tag{70}$$

where we set $\mathcal{C}_1(0)=\mathcal{C}_2(0)=\mathcal{C}_3(0)=1$. Equating coefficients of q^n on both sides of (70), we arrive at the desired result. □

Example:

Table 5. The case $n=11$ in Theorem 3.6.

$\mathcal{C}_1(11)=2$	$\mathcal{C}_2(8)=2$	$\mathcal{C}_3(11)=4$
$6+5_r$	8_r	11_r
$6+5_g$	8_g	11_g
		5_r+3+3
		5_g+3+3

Theorem 3.7. *Let* $\mathcal{C}_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm2,\pm7,\pm8$ *or* $\pm16 \pmod{32}$ *such that the parts congruent to* ±7 *and* $\pm8 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_2(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm2,\pm8,\pm9$ *or* $\pm16 \pmod{32}$ *such that parts congruent to* ±8 *and* $\pm9 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm1,\pm7,\pm9$ *or* $\pm15 \pmod{32}$ *such that the parts congruent to* ±7 *and* $\pm9 \pmod{32}$ *have 2 colours. Then for any integer* $n\geq1$,

$$\mathcal{C}_1(n)+\mathcal{C}_2(n-1)-\mathcal{C}_3(n)=0.$$

Proof. Employing (4), (6), (7) and (12) in Theorem 2.1 (ii) and employing the same procedure, we obtain

$$\frac{1}{(q^{2\pm,16\pm};q^{32})_\infty(q^{7\pm,8\pm};q^{32})^2_\infty}+\frac{q}{(q^{2\pm,16\pm};q^{32})_\infty(q^{8\pm,9\pm};q^{32})^2_\infty}-\frac{1}{(q^{1\pm,15\pm};q^{32})_\infty(q^{7\pm,9\pm};q^{32})^2_\infty}=0. \tag{71}$$

The above quotients of (71) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (71) is equivalent to

$$\sum_{n=0}^{\infty}\mathcal{C}_1(n)q^n+q\sum_{n=0}^{\infty}\mathcal{C}_2(n)q^n-\sum_{n=0}^{\infty}\mathcal{C}_3(n)q^n=0, \tag{72}$$

where we set $\mathcal{C}_1(0)=\mathcal{C}_2(0)=\mathcal{C}_3(0)=1$. Equating coefficients of q^n on both sides of (70), we arrive at the desired result. □

Example:

TABLE 6. The case $n = 9$ in Theorem 3.7.

$\mathcal{C}_1(9) = 2$	$\mathcal{C}_2(8) = 3$	$\mathcal{C}_3(9) = 5$
$7_r + 2$	8_r	9_r
$7_g + 2$	8_g	9_g
	$2+2+2+2$	$7_r + 1 + 1$
		$7_g + 1 + 1$
		$1+1+1+1+1+1+1+1+1$

Theorem 3.8. *Let* $\mathcal{C}_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 3, \pm 8, \pm 10$ *or* $\pm 16 \pmod{32}$ *such that the parts congruent to* ± 3 *and* $\pm 8 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_2(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 8, \pm 10, \pm 13$ *or* $\pm 16 \pmod{32}$ *such that parts congruent to* ± 8 *and* $\pm 13 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 3, \pm 5, \pm 11$ *or* $\pm 13 \pmod{32}$ *such that the parts congruent to* ± 3 *and* $\pm 13 \pmod{32}$ *have 2 colours. Then for any integer* $n \geq 5$,

$$\mathcal{C}_1(n) + \mathcal{C}_2(n-5) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (4), (6), (7) and (13) in Theorem 2.1 (iii) and employing the same procedure, we obtain

$$\frac{1}{(q^{10\pm,16\pm};q^{32})_\infty (q^{3\pm,8\pm};q^{32})_\infty^2} + \frac{q^5}{(q^{10\pm,16\pm};q^{32})_\infty (q^{8\pm,13\pm};q^{32})_\infty^2} - \frac{1}{(q^{5\pm,11\pm};q^{32})_\infty (q^{3\pm,13\pm};q^{32})_\infty^2} = 0. \tag{73}$$

The above quotients of (73) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (73) is equivalent to

$$\sum_{n=0}^{\infty} \mathcal{C}_1(n) q^n + q^5 \sum_{n=0}^{\infty} \mathcal{C}_2(n) q^n - \sum_{n=0}^{\infty} \mathcal{C}_3(n) q^n = 0, \tag{74}$$

where we set $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathcal{C}_3(0) = 1$. Equating coefficients of q^n on both sides of (74), we arrive at the desired result. □

Example:

TABLE 7. The case $n = 13$ in Theorem 3.8.

$\mathcal{C}_1(13) = 2$	$\mathcal{C}_2(8) = 2$	$\mathcal{C}_3(13) = 4$
$10 + 3_r$	8_r	9_r
$7_g + 2$	8_g	9_g
	$2+2+2+2$	$7_r + 1 + 1$
		$7_g + 1 + 1$
		$1+1+1+1+1+1+1+1+1$

Theorem 3.9. *Let* $\mathcal{C}_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1, \pm 8, \pm 14$ *or* $\pm 16 \pmod{32}$ *such that the parts congruent to* ± 1 *and* $\pm 8 \pmod{32}$ *have 2 colours. Let* $\mathcal{C}_2(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 8, \pm 14, \pm 15$ *or* $\pm 16 \pmod{32}$ *such that parts congruent to* ± 8 *and* $\pm 15 \pmod{32}$

have 2 colours. Let $\mathcal{C}_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm1, \pm7, \pm9$ *or* $\pm 15 \pmod{32}$ *such that the parts congruent to* ±1 *and* $\pm 15 \pmod{32}$ *have 2 colours. Then for any integer* $n \geq 7$,

$$\mathcal{C}_1(n) + \mathcal{C}_2(n-7) - \mathcal{C}_3(n) = 0.$$

Proof. Employing (4), (6), (7) and (14) in Theorem 2.1 (iv) and employing the same procedure, we obtain

$$\frac{1}{(q^{14\pm,16\pm};q^{32})_\infty (q^{1\pm,8\pm};q^{32})_\infty^2} + \frac{q^7}{(q^{14\pm,16\pm};q^{32})_\infty (q^{8\pm,15\pm};q^{32})_\infty^2} - \frac{1}{(q^{7\pm,9\pm};q^{32})_\infty (q^{1\pm,15\pm};q^{32})_\infty^2} = 0. \tag{75}$$

The above quotients of (75) represent the generating functions for $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$ and $\mathcal{C}_3(n)$, respectively. Hence, (75) is equivalent to

$$\sum_{n=0}^{\infty} \mathcal{C}_1(n) q^n + q^7 \sum_{n=0}^{\infty} \mathcal{C}_2(n) q^n - \sum_{n=0}^{\infty} \mathcal{C}_3(n) q^n = 0, \tag{76}$$

where we set $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathcal{C}_3(0) = 1$. Equating coefficients of q^n on both sides of (74), we arrive at the desired result. ☐

Example:

TABLE 8. The case $n = 8$ in Theorem 3.9.

$\mathcal{C}_1(8) = 11$	$\mathcal{C}_2(1) = 0$	$\mathcal{C}_3(8) = 11$
8_r		$7 + 1_r$
8_g		$7 + 1_g$
$1_r+1_r+1_r+1_r+1_r+1_r+1_r+1_r$		$1_r+1_r+1_r+1_r+1_r+1_r+1_r+1_r$
$1_r+1_r+1_r+1_r+1_r+1_r+1_r+1_g$		$1_r+1_r+1_r+1_r+1_r+1_r+1_r+1_g$
$1_r+1_r+1_r+1_r+1_r+1_r+1_g+1_g$		$1_r+1_r+1_r+1_r+1_r+1_r+1_g+1_g$
$1_r+1_r+1_r+1_r+1_r+1_g+1_g+1_g$		$1_r+1_r+1_r+1_r+1_r+1_g+1_g+1_g$
$1_r+1_r+1_r+1_r+1_g+1_g+1_g+1_g$		$1_r+1_r+1_r+1_r+1_g+1_g+1_g+1_g$
$1_r+1_r+1_r+1_g+1_g+1_g+1_g+1_g$		$1_r+1_r+1_r+1_g+1_g+1_g+1_g+1_g$
$1_r+1_r+1_g+1_g+1_g+1_g+1_g+1_g$		$1_r+1_r+1_g+1_g+1_g+1_g+1_g+1_g$
$1_r+1_g+1_g+1_g+1_g+1_g+1_g+1_g$		$1_r+1_g+1_g+1_g+1_g+1_g+1_g+1_g$
$1_g+1_g+1_g+1_g+1_g+1_g+1_g+1_g$		$1_g+1_g+1_g+1_g+1_g+1_g+1_g+1_g$

4. General theorems for explicit values of $J_i(q), i = 1, 2, 3, 4$

In this section, we offer general theorems to find explicit values of $J_1(q)$, $J_2(q)$, $J_3(q)$ and $J_4(q)$. Here it is useful to note the following two continued fractions of order twenty-four from [6]:

$$M(q) := q^{3/2} \frac{\mathfrak{f}(-q,-q^{15})}{\mathfrak{f}(-q^7,-q^9)} \tag{77}$$

$$= \cfrac{q^{3/2}(1-q)}{(1-q^4)+\cfrac{q^4(1-q^3)(1-q^5)}{(1-q^4)(1+q^8)+\cfrac{q^4(1-q^{11})(1-q^{13})}{(1-q^4)(1+q^{16})+\cdots}}}$$

and

$$N(q) := q^{1/2}\frac{\mathfrak{f}(-q^3,-q^{13})}{\mathfrak{f}(-q^5,-q^{11})} \tag{78}$$

$$= \cfrac{q^{1/2}(1-q^3)}{(1-q^4)+\cfrac{q^4(1-q)(1-q^7)}{(1-q^4)(1+q^8)+\cfrac{q^4(1-q^9)(1-q^{15})}{(1-q^4)(1+q^{16})+\cdots}}}.$$

Theorem 4.1. *We have*

(i) $\dfrac{1}{J_1(e^{-\pi\sqrt{n}/4})} - J_1(e^{-\pi\sqrt{n}/4}) = N(e^{-\pi\sqrt{n}/4})\Big(\dfrac{1}{H(e^{-\pi\sqrt{n}})} - H(e^{-\pi\sqrt{n}})\Big)$,

(ii) $\dfrac{1}{J_2(e^{-\pi\sqrt{n}/4})} - J_2(e^{-\pi\sqrt{n}/4}) = M(e^{-\pi\sqrt{n}/4})\Big(\dfrac{1}{H(e^{-\pi\sqrt{n}})} - H(e^{-\pi\sqrt{n}})\Big)$,

(iii) $\dfrac{1}{J_3(e^{-\pi\sqrt{n}/4})} - J_3(e^{-\pi\sqrt{n}/4}) = \dfrac{1}{N(e^{-\pi\sqrt{n}/4})}\Big(\dfrac{1}{H(e^{-\pi\sqrt{n}})} - H(e^{-\pi\sqrt{n}})\Big)$,

(iv) $\dfrac{1}{J_4(e^{-\pi\sqrt{n}/4})} - J_4(e^{-\pi\sqrt{n}/4}) = \dfrac{1}{M(e^{-\pi\sqrt{n}/4})}\Big(\dfrac{1}{H(e^{-\pi\sqrt{n}})} - H(e^{-\pi\sqrt{n}})\Big)$.

Proof. Employing (24) in (15) and then employing (9) and (78), we obtain

$$\frac{1}{J_1(q)} - J_1(q) = N(q)\left(\frac{1}{H(q^4)} - H(q^4)\right). \tag{79}$$

Setting $q = e^{-\pi\sqrt{n}/4}$ in (79), we arrive at (i). Proofs of (ii)-(iv) are follow identically from (16)-(18), respectively. □

Remark 4.2. *From Theorem 4.1, it is easily seen that to evaluate the explicit values* $J_1(e^{-\pi\sqrt{n}/4})$, $J_2(e^{-\pi\sqrt{n}/4})$, $J_3(e^{-\pi\sqrt{n}/4})$ *and* $J_4(e^{-\pi\sqrt{n}/4})$ *it is sufficient to know the values of* $M(e^{-\pi\sqrt{n}/4})$, $N(e^{-\pi\sqrt{n}/4})$ *and* $H(e^{-\pi\sqrt{n}})$. *In* [6], *authors proved some general theorems for the explicit values of* $M(q)$ *and* $N(q)$ *and evaluated some explicit values. For example, they evaluated*

$$M(e^{-\pi/4}) = \frac{1}{2^{3/8}\sqrt{1+\sqrt{2}}}\Big[2^{5/4} - \sqrt{2\left(2+2^{5/4}+2\sqrt{2}+2^{3/4}-2^{3/8}\sqrt{4+2^{9/4}+4\sqrt{2}+3\cdot 2^{3/4}}\right)}\Big] \tag{80}$$

and

$$N(e^{-\pi/4}) = \frac{1}{2}\Big[2^{5/8}\sqrt{1+\sqrt{2}} - 2^{1/4}\sqrt{4+2^{5/4}+2^{3/4}} + \sqrt{4+(2^{5/8}\sqrt{1+\sqrt{2}}-2^{1/4}\sqrt{4+2^{5/4}+2^{3/4}})^2}\Big], \tag{81}$$

Baruah and Saikia [1] *evaluated explicit values of* $H(e^{-\pi\sqrt{n}})$. *For example,*

$$H(e^{-\pi}) = \sqrt{2(2+\sqrt{2})} - 1 - \sqrt{2}. \tag{82}$$

Taking $n=1$, *employing* (81) *and* (82) *in* (*i*) *and then solving the resulting quadratic equation, we obtain*

$$\begin{aligned} J_1(e^{-\pi/4}) = \frac{1}{2}\Big[&(1+\sqrt{2})\Big(2^{1/4}x_1 - 2^{5/8}\sqrt{1+\sqrt{2}}\Big) \\ &-(2+\sqrt{2})\sqrt{2x_3 - 2^{7/8}x_2} + \Big(4 + \Big((1+\sqrt{2})\Big(2^{5/8}\sqrt{1+\sqrt{2}} - 2^{1/4}x_1\Big) \\ &+(2+\sqrt{2})\sqrt{2x_3 - 2^{7/8}x_2}\Big)^2\Big)^{1/2}\Big]. \end{aligned} \tag{83}$$

Again, employing (80) *and* (82) *in (ii) and then solving the resulting quadratic equation, we obtain*

$$\begin{aligned} J_2(e^{-\pi/4}) = \frac{1}{2}\Big[&-2\cdot 2^{7/8}\sqrt{1+\sqrt{2}} + 2\cdot 2^{1/8}\sqrt{(1+\sqrt{2})(x_4 - 2^{3/8})x_2} \\ &+\sqrt{4 + \Big(2\cdot 2^{7/8}\sqrt{1+\sqrt{2}} - 2\cdot 2^{1/8}\sqrt{(1+\sqrt{2})(x_4 - 2^{3/8}x_2)}\Big)^2}\ \Big], \end{aligned} \tag{84}$$

where

$$x_1 = \sqrt{4 + 2^{5/4} + 2^{3/4}}, \quad x_2 = \sqrt{4 + 4\cdot 2^{5/4} + 4\sqrt{2} + 3\cdot 2^{3/4}},$$

$$x_3 = 1 + 2^{1/4} + \sqrt{2} + 2^{3/4} \quad \textit{and} \quad x_4 = 2 + 2\cdot 2^{1/4} + 2\sqrt{2} + 2^{3/4}.$$

To choose the appropriate root of the quadratic equation, we used the fact that $J_1(q) \approx q^{3/2}(1-q^5)(1+q^8)$ *by neglecting terms involving* q^{16} *or higher powers of* q *as* $|q| < 1$. *Similarly, one can calculate explicit values of* $J_3(e^{-\pi/4})$ *and* $J_4(e^{-\pi/4})$ *by using Theorem 4.1 (iii) and (iv), respectively.*

5. **Vanishing coefficient results**

In this section, we obtain vanishing coefficient results from the continued fractions $J_2(q)$, $J_3(q)$, $J_4(q)$ and their reciprocals.

Theorem 5.1. *If*

$$J_2^*(q) := q^{-1/2}J_2(q) = \frac{\mathfrak{f}(-q^7,-q^{25})}{\mathfrak{f}(-q^9,-q^{23})} = \sum_{n=0}^{\infty} k_n q^n \quad \textit{and} \quad \frac{1}{J_2^*(q)} = \sum_{n=0}^{\infty} k'_n q^n\ ,$$

then

$$(i) \quad k_{16n+3} = 0 \quad \textit{and} \quad (ii) \quad k'_{16n+4} = 0.$$

Proof. Write

$$\sum_{n=0}^{\infty} k_n q^n = \frac{\mathfrak{f}(-q^7,-q^{25})}{\mathfrak{f}(-q^9,-q^{23})} = \frac{\mathfrak{f}\left(-q^7,-q^{25}\right)\mathfrak{f}\left(q^9,q^{23}\right)}{\mathfrak{f}\left(-q^9,-q^{23}\right)f\left(q^9,q^{23}\right)}. \tag{85}$$

From [2, p. 45, Entry 29], we note that, if a, b, c and d are complex numbers satisfying $ab = cd$, then

$$\mathfrak{f}(a,b)\mathfrak{f}(c,d) = \mathfrak{f}(ac,bd)\mathfrak{f}(ad,bc) + a\mathfrak{f}(b/c,ac^2d)\mathfrak{f}(b/d,acd^2). \tag{86}$$

Setting $a = -q^7, b = -q^{25}, c = q^9, d = q^{23}$ in (86), we obtain

$$\mathfrak{f}(-q^7,-q^{25})\mathfrak{f}(q^9,q^{23}) = \mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{30},-q^{34}) \tag{87}$$

$$-q^7\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^2,-q^{62}).$$

Again, from [2, p. 46, Entry 30 (iv)], we note that

$$\mathfrak{f}(a,b)\mathfrak{f}(-a,-b)=\mathfrak{f}(-a^2,-b^2)\phi(-ab). \tag{88}$$

Setting $a=q^9$ and $b=q^{23}$ in (88), we obtain

$$\mathfrak{f}(q^9,q^{23})\mathfrak{f}(-q^9,-q^{23})=\mathfrak{f}(-q^{18},-q^{46})\phi(-q^{32}). \tag{89}$$

Employing (87) and (89) in (85), we obtain

$$\sum_{n=0}^{\infty}k_nq^n=\frac{\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^{30},-q^{34})-q^7\mathfrak{f}(-q^{16},-q^{48})\mathfrak{f}(-q^2,-q^{62})}{\mathfrak{f}(-q^{18},-q^{46})\phi(-q^{32})}. \tag{90}$$

Extracting the terms involving q^{2n+1} in (90), dividing by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}k_{2n+1}q^n&=-q^3\frac{\mathfrak{f}(-q^8,-q^{24})\mathfrak{f}(-q,-q^{31})}{\mathfrak{f}(-q^9,-q^{23})\phi(-q^{16})}\\&=-q^3\frac{\mathfrak{f}(-q^8,-q^{24})\mathfrak{f}(-q,-q^{31})\mathfrak{f}(q^9,q^{23})}{\mathfrak{f}(-q^9,-q^{23})\mathfrak{f}(q^9,q^{23})\phi(-q^{16})}.\end{aligned} \tag{91}$$

Setting $a=-q$, $b=-q^{31}$, $c=q^9$ and $d=q^{23}$ in (86), we obtain

$$\begin{aligned}\mathfrak{f}(-q,-q^{31})\mathfrak{f}(q^9,q^{23})&=\mathfrak{f}(-q^{10},-q^{54})\mathfrak{f}(-q^{24},-q^{40})\\&\quad-q\mathfrak{f}(-q^{22},-q^{42})\mathfrak{f}(-q^8,-q^{56}).\end{aligned} \tag{92}$$

Applying (92) and (89) in (91), we have

$$\begin{aligned}\sum_{n=0}^{\infty}k_{2n+1}q^n=-q^3\frac{\mathfrak{f}(-q^8,-q^{24})}{\mathfrak{f}(-q^{18},-q^{46})\phi(-q^{16})\phi(-q^{32})}\Big\{&\mathfrak{f}(-q^{10},-q^{54})\mathfrak{f}(-q^{24},-q^{40})\\&-q\mathfrak{f}(-q^{22},-q^{42})\mathfrak{f}(-q^8,-q^{56})\Big\}.\end{aligned} \tag{93}$$

Again, extracting the terms involving q^{2n+1} in (93), dividing by q and replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}k_{4n+3}q^n&=-q\frac{\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^5,-q^{27})\mathfrak{f}(-q^{12},-q^{20})}{\mathfrak{f}(-q^9,-q^{23})\phi(-q^8)\phi(-q^{16})}\\&=-q\frac{\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^{12},-q^{20})\mathfrak{f}(-q^5,-q^{27})\mathfrak{f}(q^9,q^{23})}{\mathfrak{f}(-q^9,-q^{23})\mathfrak{f}(q^9,q^{23})\phi(-q^8)\phi(-q^{16})}.\end{aligned} \tag{94}$$

Setting $a=-q^5$, $b=-q^{27}$, $c=q^9$ and $d=q^{23}$ in (86), we obtain

$$\begin{aligned}\mathfrak{f}(-q^5,-q^{27})\mathfrak{f}(q^9,q^{23})&=\mathfrak{f}(-q^{14},-q^{50})\mathfrak{f}(-q^{28},-q^{36})\\&\quad-q^5\mathfrak{f}(-q^{18},-q^{46})\mathfrak{f}(-q^4,-q^{60}).\end{aligned} \tag{95}$$

Applying (95) and (89) in (94), we have

$$\begin{aligned}\sum_{n=0}^{\infty}k_{4n+3}q^n=\frac{-q\mathfrak{f}(-q^4,-q^{12})\mathfrak{f}(-q^{12},-q^{20})}{\mathfrak{f}(-q^{18},-q^{46})\phi(-q^8)\phi(-q^{16})\phi(-q^{32})}\Big\{&\mathfrak{f}(-q^{14},-q^{50})\\ \mathfrak{f}(-q^{28},-q^{36})-q^5\mathfrak{f}(-q^{18},-q^{46})\mathfrak{f}(-q^4,-q^{60})\Big\}.\end{aligned} \tag{96}$$

Again, extracting the terms involving q^{2n} and replacing q^2 by q in (96), we obtain

$$\sum_{n=0}^{\infty} k_{8n+3} q^n = q^3 \frac{\mathfrak{f}(-q^2,-q^6)\mathfrak{f}(-q^6,-q^{10})\mathfrak{f}(-q^2,-q^{30})}{\phi(-q^4)\phi(-q^8)\phi(-q^{16})}. \tag{97}$$

The right hand side of (97) contains no term involving q^{2n}, so extracting terms involving q^{2n} and replacing q^2 by q, we arrive at (i). Similarly, we obtain (ii). □

In next theorems, we offer vanishing coefficients arising from the continued fractions $J_3(q)$ and $J_4(q)$. Since the proofs are identical to the proof of Theorem 5.1, we only state the results and omit proofs.

Theorem 5.2. *If*

$$J_3^*(q) := q^{-5/2} J_3(q) = \frac{\mathfrak{f}(-q^3,-q^{29})}{\mathfrak{f}(-q^{13},-q^{19})} = \sum_{n=0}^{\infty} h_n q^n \quad \text{and} \quad \frac{1}{J_3^*(q)} = \sum_{n=0}^{\infty} h'_n q^n,$$

then

$$h_{16n+5} = 0, \quad \text{and} \quad h'_{16n+10} = 0.$$

Theorem 5.3. *If*

$$J_4^*(q) := q^{-7/2} J_4(q) = \frac{\mathfrak{f}(-q,-q^{31})}{\mathfrak{f}(-q^{15},-q^{17})} = \sum_{n=0}^{\infty} g_n q^n \quad \text{and} \quad \frac{1}{J_4^*(q)} = \sum_{n=0}^{\infty} g'_n q^n,$$

then

$$g_{16n+8} = 0, \quad \text{and} \quad g'_{16n+15} = 0.$$

Acknowledgement

The authors are grateful to the anonymous referee, who read our manuscript with great care and offered useful suggestions. The first author acknowledges the financial support received from the Department of Science and Technology (DST), Government of India through INSPIRE Fellowship [DST/INSPIRE Fellowship/2021/IF210210].

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